Linear Programming

- Used to obtain optimal solutions to problems that involve restrictions or limitations, such as optimizing profit (or costs or time or …) subject to availability of:
  - Materials
  - Budgets
  - Labor
  - Machine time

Linear Programming

- **Linear programming** (LP) techniques consist of a sequence of steps that will (eventually) lead to an optimal solution(s) to problems
  - in cases where an optimum exists (problem might be infeasible)

Linear Programming Model

- **Objective**: the goal of an LP model is maximization or minimization (e.g. minimize cost, max profit, …)
- **Decision variables**: amounts of either inputs or outputs (e.g. # of people to hire/fire, products to build, …)
- **Feasible solution space**: the set of all feasible combinations of decision variables as defined by the constraints
- **Constraints**: limitations that restrict the available alternatives (e.g. available money, time, supplies, …)
- **Parameters**: numerical values (e.g. upper limits)

Linear Programming Assumptions

- **Linearity**: the impact of decision variables is linear in constraints and objective function (hence the name linear programming)
  - $5x - 2y > 10$ vs. $5xy > 22$
- **Divisibility**: are non-integer values of decision variables acceptable? (you need to specify this assumption)
- **Certainty**: values of parameters are known and constant (vs. probabilistic parameters such as future demand)
  - Sensitivity analysis helps understanding of solution certainty
- **Nonnegativity**: are negative values of decision variables unacceptable? (you need to specify this)
**Linear Programming Example**

- Objective: profit per product $x_1$ & $x_2$ is:
  
  \[
  \text{Maximize } Z = 60x_1 + 50x_2
  \]

- Subject to
  
  - Assembly: $4x_1 + 10x_2 \leq 100$ hours
  - Inspection: $2x_1 + 1x_2 \leq 22$ hours
  - Storage: $3x_1 + 3x_2 \leq 39$ cubic feet
  
  $x_1, x_2 \geq 0$

Some basic data represented in this model:
- What is profit on each unit of $x_1$?
- How many hours does it take to assemble $x_2$?
- How many feet of storage does $x_1$ require?
- How many inspection hours are available?

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**Graphical Linear Programming**

1. Set up objective function and constraints in mathematical format
2. Plot the constraints
3. Identify the feasible solution space
4. Plot the objective function
5. Determine the optimum solution

*The graphical method is used for instructional purposes only (not in real life) in order to present basic LP foundations, but we will not be spending much time at all on it, preferring instead to go straight to Excel solutions.*
**Solution**

- The intersection of inspection and storage
- Solve two equations in two unknowns
  \[ 2X_1 + 1X_2 = 22 \]
  \[ 3X_1 + 3X_2 = 39 \]

  \[ X_1 = 9 \]
  \[ X_2 = 4 \]
  \[ Z = 740 \]

**Constraints**

- **Redundant constraint**: a constraint that does not form a unique boundary of the feasible solution space
  - i.e. it is redundant & does not affect the solution because other constraint(s) are more critical
- **Binding constraint**: a constraint that forms the optimal corner point of the feasible solution space
  - i.e. this constraint is restricting the solution from being even more optimal (so having more of this constraint would be better – such as more labor, time, material, etc.)

**Slack and Surplus**

- **Surplus**: when the optimal values of decision variables are substituted into a greater than or equal to constraint and the resulting value exceeds the right side value
  - E.g. \((2)(5) + 3(2) > 10 \) ➔ a surplus of 
- **Slack**: when the optimal values of decision variables are substituted into a less than or equal to constraint and the resulting value is less than the right side value
  - E.g. \((2)(5) - 3(2) < 5 \) ➔ a slack of 

**Simplex Method**

- **Simplex**: a linear-programming algorithm that can solve problems having many more than two decision variables
  - i.e. the typical situation
  - Excel’s Solver uses the simplex method

**Figure 6S.15**

**Figure 6S.17**

**MS Excel Worksheet for Microcomputer Problem**

**MS Excel Worksheet Solution**
**Sensitivity Analysis**

- **Range of optimality**: the range of values for which the solution quantities of the decision variables remains the same
- **Range of feasibility**: the range of values for the right-hand side of a constraint over which the shadow price remains the same
- **Shadow prices**: negative values indicating how much a one-unit decrease in the original amount of a constraint would decrease the final value of the objective function

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**Linear Programming in Excel - Part 1**

One of Excel’s useful features is the ability to solve linear programming problems (especially those beyond our graphical abilities).

The feature is invoked by creating a spreadsheet containing the objective function and constraints, selecting Tools from the main menu, and Solver from the submenu.

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**Linear Programming in Excel - Part 2**

Here is an Excel version of the moonshine problem.

<table>
<thead>
<tr>
<th></th>
<th>White Lightning</th>
<th>Rotgut</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corn (bushels)</td>
<td>$=60^*C52+D60^*D52 &lt;= 40$</td>
<td></td>
</tr>
<tr>
<td>Sugar (pounds)</td>
<td>$=60^*C52+D60^*D52 &lt;= 70$</td>
<td></td>
</tr>
<tr>
<td>Jugs</td>
<td>$=60^*C52+D60^*D52 &lt;= 50$</td>
<td></td>
</tr>
<tr>
<td>Hours</td>
<td>$=60^*C52+D60^*D52 &lt;= 72$</td>
<td></td>
</tr>
<tr>
<td>$12$</td>
<td>$10$</td>
<td></td>
</tr>
<tr>
<td>$8$</td>
<td>$20$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$12$</td>
<td>$8$</td>
</tr>
<tr>
<td>$10$</td>
<td>$20$</td>
</tr>
<tr>
<td>$280.00$</td>
<td>Profit</td>
</tr>
</tbody>
</table>

The top view shows formulas and the bottom view shows initial calculations.

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**Linear Programming in Excel - Part 3**

Once the basic set of equations has been entered, launch Solver and fill in the dialog boxes with references to your sheet.

- **Target cell**: The objective function value (E52)
- **Equal to**: Choose max or min based on the problem
- **By changing cells**: The decision variables (C52:D52)
- **Subject to constraints**: Add all constraints one at a time by referencing their function values (e.g., the amount of corn used, E47 must be less than the amount of corn on hand, G47)

Once all constraints have been entered, choose Options and check the boxes for Assume Linear Model and Assume Non-Negative.

Finally, choose Solve and wait for Excel to work its magic.

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**Solver Output Reports - Part 1**

- **Answer Report**: contains the basic answer to the problem and reveals which constraints had an impact on your situation.
- **Sensitivity Report**: tells you reduced costs and shadow prices
- **Limits Report**: don’t bother asking for this one. We won’t use its information.
Target Cell (Max)

<table>
<thead>
<tr>
<th>Cell Name</th>
<th>Original Value</th>
<th>Final Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$52</td>
<td>Profit</td>
<td>$20.00 $280.00</td>
</tr>
</tbody>
</table>

Adjustable Cells

<table>
<thead>
<tr>
<th>Cell Name</th>
<th>Original Value</th>
<th>Final Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$52</td>
<td>White Lightning 1</td>
<td>10</td>
</tr>
<tr>
<td>$D$52</td>
<td>Rotgut</td>
<td>20</td>
</tr>
</tbody>
</table>

Constraints

<table>
<thead>
<tr>
<th>Cell Name</th>
<th>Cell Value</th>
<th>Formula</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$47</td>
<td>Corn</td>
<td>40</td>
<td>$E$47&lt;=$F$47 Binding</td>
</tr>
<tr>
<td>$E$48</td>
<td>Sugar</td>
<td>70</td>
<td>$E$48&lt;=$F$48 Binding</td>
</tr>
<tr>
<td>$E$49</td>
<td>Jugs</td>
<td>30</td>
<td>$E$49&lt;=$F$49 Not Binding</td>
</tr>
<tr>
<td>$E$50</td>
<td>Hours</td>
<td>70</td>
<td>$E$50&lt;=$F$50 Not Binding</td>
</tr>
</tbody>
</table>

The value of the objective function at the optimal solution

The optimal values of the decision variables

If a constraint is not binding, then we have some slack (left over) when we implement the optimal solution. We have 20 extra jugs and 2 hours to spare.

A binding constraint is one that limits the value our objective function can assume. We use up all of our corn and sugar (no slack).

Extra amount of resource needed for a binding constraint to become non-binding. We have 20 extra jugs and 2 hours to spare.

A one unit decrease in the original amount of sugar available will decrease our profit by this amount.

Increases or decreases within these ranges will result in the same product mix (but a different objective function value).

Solver Output Reports – Part 2

Adjustable Cells

<table>
<thead>
<tr>
<th>Cell Name</th>
<th>Final Value</th>
<th>Reduced Objective Cost</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$52</td>
<td>White</td>
<td>10</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$D$52</td>
<td>Rotgut</td>
<td>20</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Constraints

<table>
<thead>
<tr>
<th>Cell Name</th>
<th>Final Value</th>
<th>Shadow Price</th>
<th>R.H. Side</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$47</td>
<td>Corn</td>
<td>40</td>
<td>-</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$E$48</td>
<td>Sugar</td>
<td>70</td>
<td>4</td>
<td>70</td>
<td>10</td>
</tr>
<tr>
<td>$E$49</td>
<td>Jugs</td>
<td>30</td>
<td>-</td>
<td>50</td>
<td>20</td>
</tr>
<tr>
<td>$E$50</td>
<td>Hours</td>
<td>70</td>
<td>-</td>
<td>72</td>
<td>18+30</td>
</tr>
</tbody>
</table>

A one unit decrease in the original amount of sugar available will decrease our profit by this amount.

Extra amount of resource needed for a binding constraint to become non-binding.

Amount of resource to be taken away for a non-binding constraint to become binding.

Note that this doesn’t apply to non-binding constraints, hence the huge amounts indicated.